

On κ -Deformed $D = 4$ Quantum Conformal Group^{*}

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Abstract

This paper is presented on the occasion of 60-th birthday of Jose Adolfo de Azcarraga who in his very rich scientific curriculum vitae has also a chapter devoted to studies of quantum-deformed symmetries, in particular deformations of relativistic and Galilean space-time symmetries [1]–[4].

In this paper we provide new steps toward describing the κ -deformed $D = 4$ conformal group transformations. We consider the quantization of $D = 4$ conformal group with dimensionful deformation parameter κ . Firstly we discuss the noncommutativity following from the Lie-Poisson structure described by the light-cone κ -Poincaré r -matrix. We present complete set of $D = 4$ conformal Lie-Poisson brackets and discuss their quantization. Further we define the light-cone κ -Poincaré quantum R -matrix in $O(4, 2)$ vector representation and discuss the inclusion of non-commutative conformal translations into the framework of κ -deformed conformal quantum group. The problem with real structure of κ -deformed conformal group is pointed out.

1 Introduction

The standard κ -deformation obtained in 1991–92 [5]–[7] leads to the introduction of quantum time coordinate. Indeed, after introducing the dual pair of

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standard κ -deformed Poincaré algebra and κ -deformed Poincaré group [8]–[11] we get the κ -deformed Minkowski space $\hat{x}_\mu = (\hat{x}_i, \hat{x}_0)$ described by the following algebraic relations.

$$[\hat{x}_i, \hat{x}_j] = 0, \quad [\hat{x}_0, \hat{x}_i] = \frac{i}{\kappa} x_i. \quad (1.1)$$

Further in 1995–96 there was introduced the generalized κ -deformation [12]–[14], with quantized direction $y_0 = a^\mu x_\mu$ in Minkowski space, where a_μ is an arbitrary constant fourvector. The relations (1.1) were replaced by the following a_μ -dependent commutator [12]–[16]

$$[\hat{x}_\mu, \hat{x}_\nu] = \frac{1}{\chi} (a^\mu \hat{x}^\nu - a^\nu \hat{x}^\mu). \quad (1.2)$$

In particular if $a_\mu = (1, 0, 0, 0)$ we obtain from (1.2) the relations (1.1).

It appears that if a_μ is light-like ($a_\mu^2 = 0$) the classical r -matrix corresponding to the relation (1.2) satisfies classical Yang–Baxter equation. We call such a deformation the light-cone quantum κ -deformation of Poincaré symmetries. It has been shown [17] that after nonlinear change of basis the light-cone κ -deformation of Poincaré algebra can be identified with the null-plane quantum deformation proposed independently by Ballesteros et al [18].

The classical r -matrix (1.2) if $a_\mu^2 = 0$ can be also used as the classical r -matrix describing the quantum deformation of $D = 4$ Weyl and $D = 4$ conformal symmetries. It should be recalled that the light-cone κ -deformation of Poincaré algebra has been extended in [12] to the deformation of $D = 4$ Weyl algebra containing besides the Poincaré generators $(P_\mu, M_{\mu\nu})$ also the dilatation generator D . Recently [15] there was also employed the light-cone κ -deformation of $D = 4$ Poincaré algebra in the framework of twist quantization technique [19, 20] as a member of three-parameter family of quantum deformations of $D = 4$ conformal algebra.

In this paper we plan to study the κ -deformation of $D = 4$ conformal group, generated by the light-cone κ -Poincaré r -matrix. In [8] it has been shown that the standard κ -deformation of Poincaré group can be obtained by quantization of the Lie-Poisson bracket. Such a method has been extended to the description of noncommutative symmetry parameters for the $D = 4$ Weyl group [12] as well as for $D = 4$ Poincaré supergroup [21]. Indeed, introducing the quantum R -matrix and applying the FRT method [22] describing noncommutativity by means of “RTT=TTR equations” one can show that quantized Lie-Poisson brackets provide for the cases of light-cone κ -Poincaré group, Weyl group and Poincaré supergroup the transition from classical to quantum group symmetries. Unfortunately, due to higher nonlinearities of Lie-Poisson brackets for conformal translations, such a method does not work in straightforward way for the κ -deformed conformal group.

In this paper we consider firstly in Sect. 2 the noncommutativity of all conformal group parameters obtained from the Lie-Poisson structure determined by the light-cone κ -Poincaré r -matrix. It appears that in such a way in the presence of conformal translations we obtain only leading terms (of order $\frac{1}{\kappa}$)

describing the noncommutativity of quantum group parameters. In order to compare these results with the complete set of quantum group relations, given by FRT method [22] providing the noncommutativity in all orders, we define in Sect. 3 the set of “RTT = TTR equations” for the case of κ -deformed conformal group. We present explicit formulae for the conformal quantum symmetry parameters which span the coset $\frac{SU(2,2)}{SL(2,C)}$ (translations, conformal translations and dilatations). In such a framework however we were not able to impose the reality conditions for quantized conformal group parameters. We would like to recall here that the consistency of quantum deformations of $D = 4$ conformal symmetries with reality structure (\ast -Hopf algebra structure) is also a nontrivial problem in the discussion of Drinfeld-Jimbo q -deformation of $SU(2, 2) \simeq O(4, 2)$ conformal Lie algebra (see e.g. [23]).

Finally it should be mentioned that recently a κ -deformation of $D = 4$ conformal algebra has been studied [24, 25] as generated by the Jordanian κ -deformation [26] of the $O(2, 1)$ subalgebra (D, P_0, K_0) of $D = 4$ conformal algebra. In the paper [25] there was also pointed out the incompleteness of quantization procedure of conformal Lie-Poisson bracket relations. The complete quantization was only performed in [25] for the Weyl subgroup, and even in this case of simpler deformation the problem of real (\ast -Hopf algebra) deformation of full $D = 4$ conformal group has not been resolved.

2 Lie-Poisson Structure on $D = 4$ Conformal Group and its Quantization

We consider firstly the classical conformal group acting on generalized space-time with symmetric tensor $g_{\mu\nu}$. The general conformal transformations on Minkowski space vector $u^\mu = (y^i, y^0)$ look as follows:

$$y^\mu = e^d \Lambda^\mu_\nu \frac{y^\nu + c^\nu y^2}{1 + 2cy + c^2 y^2} + a^\mu, \quad (2.1)$$

where a^μ , d , c^μ and Λ^ν_μ are respectively translations, dilatations, conformal translations and Lorentz group parameters.

The relevant composition law,

$$(a', c', d', \Lambda') * (a, c, d, \Lambda) = (a'', c'', d'', \Lambda''), \quad (2.2a)$$

reads

$$\Lambda''^\mu{}_\nu = \Lambda'^\mu{}_\rho \tilde{\Lambda}^\rho{}_\sigma(a, c) \Lambda^\sigma{}_\nu, \quad (2.2b)$$

$$a''^\mu = a'^\mu + e^{d'} \Lambda'^\mu{}_\nu \frac{a^\nu + c'^\nu a^2}{1 + 2c'a + c'^2 a^2}, \quad (2.2c)$$

$$c''^\mu = c'^\mu + e^d \Lambda^\mu{}_\sigma \frac{c'^\sigma + a^\sigma c'^2}{1 + 2c'a + c'^2 a^2}, \quad (2.2d)$$

$$d'' = d + d' - \ln(1 + 2c'a + c'^2 a^2) , \quad (2.2e)$$

where

$$\begin{aligned} \tilde{\Lambda}^\mu_\nu(a, c) = & \delta^\mu_\nu + \frac{2}{1 + 2c'a + c'^2 a^2} (-a^\mu c'_\nu + (1 + 2c'a)c'^\mu a_\nu \\ & - c'^2 a^\mu a_\nu - a^2 c'^\mu c'_\nu) . \end{aligned} \quad (2.3)$$

It is straightforward to compute the left- and right-invariant vector fields where $M_{\mu\nu}$ and P_μ are Lorentz and translations generators respectively. We obtain

i) left-invariant fields:

$$\begin{aligned} P_\mu = & -2c_\mu \frac{\partial}{\partial d} + 2c^\rho \left(\Lambda^\alpha_\rho \frac{\partial}{\partial \Lambda^{\alpha\mu}} - \Lambda^\alpha_\mu \frac{\partial}{\partial \Lambda^{\alpha\rho}} \right) \\ & + c^2 \frac{\partial}{\partial c^\mu} - 2c_\mu c^\alpha \frac{\partial}{\partial c^\alpha} + e^d \Lambda^\alpha_\mu \frac{\partial}{\partial a^\alpha} \\ M_{\mu\nu} = & \Lambda^\alpha_\mu \frac{\partial}{\partial \Lambda^{\alpha\nu}} - \Lambda^\alpha_\nu \frac{\partial}{\partial \Lambda^{\alpha\mu}} + \left(c_\mu \frac{\partial}{\partial c^\nu} - c_\nu \frac{\partial}{\partial c^\mu} \right) \end{aligned} \quad (2.4)$$

ii) right-invariant fields:

$$\begin{aligned} \tilde{P}_\mu = & \frac{\partial}{\partial a^\mu} \\ \tilde{M}_{\mu\nu} = & \Lambda_{\nu\beta} \frac{\partial}{\partial \Lambda^\mu_\beta} - \Lambda_{\mu\beta} \frac{\partial}{\partial \Lambda^\nu_\beta} + a_\nu \frac{\partial}{\partial a^\mu} - a_\mu \frac{\partial}{\partial a^\nu} . \end{aligned} \quad (2.5)$$

In order to define the Lie-Poisson structure we select the following classical r -matrix for the Poincaré group in the generalized space-time with metric $g^{\mu\nu}$

$$r = \frac{1}{\kappa} M_{0\nu} \wedge P^\nu \quad (2.6)$$

where $P^\nu = g^{\nu\rho} P_\rho$. The Schouten bracket reads

$$[r, r] = \frac{g_{00}}{\kappa^2} M_{\mu\nu} \wedge P^\mu \wedge P^\nu . \quad (2.7)$$

It should be recalled that the generalized space-time variables y_μ can be related with physical Minkowski coordinates with metric $\eta_{\mu\nu} = \text{diag}(1, -1, -1, -1)$ as follows (see e.g. [16])

$$x_\mu = R_\mu^\nu y_\nu , \quad (2.8a)$$

where from the relation $y_\mu g^{\mu\nu} y_\nu = x_\mu \eta^{\mu\nu} x_\nu$ we get that

$$R_\nu^\mu \eta^{\nu\tau} R_\tau^\rho = g^{\mu\rho} , \quad R_\nu^\mu = (R_\mu^\nu)^T \quad (2.8b)$$

Introducing $a^\mu = R_0^\mu$ one obtains for the physical Minkowski space symmetries the following formula for the classical r -matrix (2.5)

$$r = a^\mu M_{\mu\nu} \wedge P^\nu. \quad (2.9)$$

Because from (2.8a-2.8b) follows that $g^{00} = a^\mu a_\mu$ we obtain that the Schouten bracket (2.7) vanishes if $a^\mu a_\mu = 0$. In such a case (e.g. if we choose $a_\mu = (1, 1, 0, 0)$) the classical r -matrix (2.9) satisfies the classical YB equation and describes the quantum deformation of any group containing Poincaré group as its subgroup. In particular such a classical r -matrix has been used in [12] to construct the quantum version of κ -Weyl group

Below we shall consider quantum deformation of conformal group. Using Eqs. (2.4–2.5) we compute in a standard way the basic Poisson brackets:

$$\begin{aligned} \{\Lambda^\varepsilon_\lambda, \Lambda^\tau_\rho\} &= -\frac{1}{\kappa} (2c^\rho \Lambda^\tau_\rho (\Lambda^\varepsilon_0 g_{\lambda\delta} - \Lambda^\varepsilon_\delta g_{\lambda 0}) \\ &\quad - 2c_\delta (\Lambda^\varepsilon_0 \Lambda^\tau_\lambda - g_{\lambda 0} g^{\varepsilon\tau}) \\ &\quad - 2c^\rho \Lambda^\varepsilon_\rho (\Lambda^\tau_0 g_{\lambda\delta} - \Lambda^\tau_\lambda g_{\delta 0}) \\ &\quad + 2c_\lambda (\Lambda^\tau_0 \Lambda^\varepsilon_\delta - g_{\delta 0} g^{\varepsilon\tau})) \end{aligned} \quad (2.10a)$$

$$\{\Lambda^\varepsilon_\lambda, a^\tau\} = -\frac{1}{\kappa} (\Lambda^\tau_\lambda (e^d \Lambda^\varepsilon_0 - \delta^\varepsilon_0) + g^{\varepsilon\tau} (\Lambda_{0\lambda} - e^d g_{0\lambda})) \quad (2.10b)$$

$$\begin{aligned} \{\Lambda^\varepsilon_\lambda, c^\tau\} &= -\frac{1}{\kappa} (c^2 (\Lambda^\varepsilon_0 \delta^\tau_\lambda - \Lambda^{\varepsilon\tau} g_{\lambda 0}) \\ &\quad - 2c^\tau c_\lambda \Lambda^\varepsilon_0 + 2\Lambda^\varepsilon_\rho c^\rho (c^\rho g_{\lambda 0} - c_0 \delta^\tau_\lambda) \\ &\quad + 2c_0 c_\lambda \Lambda^{\varepsilon\tau}) \end{aligned} \quad (2.10c)$$

$$\{\Lambda^\varepsilon_\lambda, d\} = -\frac{2}{\kappa} (\Lambda^\varepsilon_\rho c^\rho g_{\lambda 0} - \Lambda^\varepsilon_0 c_\lambda) \quad (2.10d)$$

$$\{a^\varepsilon, a^\tau\} = -\frac{1}{\kappa} (-\delta^\varepsilon_0 a^\tau + \delta^\tau_0 a^\varepsilon) \quad (2.10e)$$

$$\{a^\varepsilon, c^\tau\} = -\frac{1}{\kappa} e^d (\Lambda^{\varepsilon\tau} c_0 - \delta^\tau_0 \Lambda^\varepsilon_\rho c^\rho) \quad (2.10f)$$

$$\{a^\varepsilon, d\} = 0 \quad (2.10g)$$

$$\{c^\varepsilon, c^\tau\} = -\frac{1}{\kappa} c^2 (c^\tau \delta^\varepsilon_0 - c^\varepsilon \delta^\tau_0) \quad (2.10h)$$

$$\{c^\varepsilon, d\} = -\frac{2}{\kappa} (c^2 \delta^\varepsilon_0 - c_0 c^\varepsilon). \quad (2.10i)$$

Eqs. (2.10a-2.10i) define Lie-Poisson structure on conformal group. One can pose the question whether this structure can be quantized by a “naive” correspondence principle $\{.,.\} \longrightarrow \frac{1}{i\hbar}[,.]$ in order to obtain a consistent Hopf algebra structure. By inspecting the terms involving conformal translations we see that after passing by “naive” quantization procedure from LP brackets to commutators we face the difficulties due to the lack of unique ordering prescription of nonlinear terms. This difficulty disappears if we consider only the Weyl group. In such a case the “naive” quantization of the LP brackets provides in straight-forward way proper Hopf algebra structure called κ -Weyl group [12].

3 FRT Quantization Technique of the Conformal Poisson-Lie Brackets

It has been shown in [13] that the κ -Poincaré algebra for $g_{00} = 0$ case, defined in [12], is isomorphic to the one introduced by Ballesteros et al [18]. However, for the latter deformation the quantum R -matrix is known [27]. This opens the way to use the FRT approach. Further we shall view therefore the conformal group (see e.g. [28]) as the matrix group $SO(4, 2)$. It is well known that the isomorphism between $SO(4, 2)$ and the conformal group can be described as follows.

Let us take the matrix tensor g_{AB} , $A, B = 0, \dots, 5$ in the form

$$g_{AB} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad (3.1)$$

and consider the cone

$$g_{AB}\xi^A\xi^B = 0. \quad (3.2)$$

Defining the homogeneous functions on this cone by $(\xi^4 + \xi^5 \neq 0)$

$$x^\mu \equiv \frac{\xi^\mu}{\xi^4 + \xi^5}, \quad \mu = 0, \dots, 3 \quad (3.3)$$

we find how $SO(4, 2)$ acts on x^μ as conformal group.

Keeping in mind the explicit action (2.1) of conformal group on x^μ and using (3.1-3.3) one can find the parametrization of general $SO(4, 2)$ matrix in terms of conformal transformations.

The general element can be written in the following 6×6 matrix form as the following product $(\mu, \nu = 0, 1, 2, 3; A, B = 0, 1, 2, 3, 4, 5)$:

$$G = T \bullet D \bullet L \bullet C. \quad (3.4)$$

The general formulae for the factors in the decomposition (3.4) looks as follows:

i) Translations (parameters a_μ)

$$\begin{aligned} T_B^A &= \delta_B^A + a^\mu \delta_\mu^A (\delta_B^4 + \delta_B^5) + a^\mu g_{\mu B} (\delta_4^A - \delta_5^A) \\ &\quad + \frac{1}{2} a^2 (\delta_4^A - \delta_5^A) (\delta_B^4 + \delta_B^5) \end{aligned} \quad (3.5a)$$

ii) Dilatations (parameter λ)

$$P_B^A = ch\lambda (\delta_5^A \delta_B^5 + \delta_4^A \delta_B^4) - sh\lambda (\delta_5^A \delta_B^4 + \delta_4^A \delta_B^5) \quad (3.5b)$$

iii) Conformal transformations (parameters c_μ)

$$\begin{aligned} C_B^A &= \delta_B^A - c^\mu \delta_\mu^A (\delta_B^5 - \delta_B^4) + c^\mu \delta_{\mu\beta} (\delta_4^A + \delta_5^A) \\ &\quad - \frac{1}{2} c^2 (\delta_4^A + \delta_5^A) (\delta_B^5 - \delta_B^4) \end{aligned} \quad (3.5c)$$

iv) Lorentz transformations (parameters Λ_B^A)

$$\begin{aligned} L_B^A &= \delta_4^A \delta_B^4 + \delta_5^A \delta_B^5 \\ &\quad + (1 - \delta_4^A) (1 - \delta_5^A) (1 - \delta_B^4) (1 - \delta_B^5) \Lambda_B^A, \end{aligned} \quad (3.5d)$$

where Λ_ν^μ is the Lorentz matrix corresponding to the $g_{00} = 0$ case.

Using this parametrization we write down the classical coproducts which are equivalent to the composition law (2.2) with an appropriate ordering:

$$\begin{aligned} \Delta(a^\mu) &= a^\mu \otimes 1 + (e^d \Lambda_\nu^\mu \otimes 1) (1 \otimes a^\nu + c^\nu \otimes a^2) \\ &\quad \cdot (1 \otimes 1 + 2c_\rho \otimes a^\rho + c^2 \otimes a^2)^{-1} \\ \Delta(e^{-d}) &= (e^{-d} \otimes e^{-d}) (1 \otimes 1 + 2c_\rho \otimes a^\rho + c^2 \otimes a^2) \\ \Delta(c^\mu) &= 1 \otimes c^\mu + (1 \otimes 1 + 2c^\nu \otimes a_\nu + c^2 \otimes a^2)^{-1} \\ &\quad \cdot (c^\beta \otimes 1 + c^2 \otimes a^\beta) (1 \otimes c^d \Lambda_\beta^\mu) \\ \Delta(\Lambda_\nu^\mu) &= \Lambda_\alpha^\mu \otimes \Lambda_\nu^\alpha + 2 (\Lambda_\alpha^\mu c^\alpha \otimes a_\beta \Lambda_\nu^\beta) \\ &\quad - 2 (\Lambda_\alpha^\mu \otimes 1) (c^\alpha \otimes a^2 + 1 \otimes a^\alpha) (1 \otimes 1 + 2c_\rho \otimes a^\rho + c^2 \otimes a^2)^{-1} \\ &\quad \cdot (c_\beta \otimes 1 + c^2 \otimes a_\beta) (1 \otimes \Lambda_\nu^\beta) \end{aligned} \quad (3.6a)$$

Now we introduce noncommuting conformal group parameters. According to Ref. [27] the quantum R -matrix is obtained by the following exponentiation procedure

$$R = e^{\frac{i}{\kappa} M^{32} \otimes P_2} e^{\frac{i}{\kappa} M^{31} \otimes P_1} e^{-\frac{i}{\kappa} P_0 \otimes M^{30}} e^{\frac{i}{\kappa} M^{30} \otimes P_0} \cdot e^{-\frac{i}{\kappa} P_1 \otimes M^{31}} e^{-\frac{i}{\kappa} P_2 \otimes M^{32}}. \quad (3.7)$$

The computation of matrix realization of (3.7) is simplified due to the nilpotency of translation generators. We obtain

$$R = I \otimes I + \frac{i}{\kappa} R_1 + \frac{i}{\kappa^2} R_2, \quad (3.8)$$

where

$$\begin{aligned} (R_1)^A{}_B{}^C{}_D &= [(\delta_4^A - \delta_5^A) \delta_B^2 - \delta_A^2 (\delta_B^4 + \delta_B^5)] (\delta_0^C \delta_D^2 + \delta_2^C \delta_D^3) \\ &+ [(\delta_4^A - \delta_5^A) \delta_B^1 - \delta_1^A (\delta_B^4 + \delta_B^5)] (\delta_0^C \delta_D^1 + \delta_1^C \delta_D^3) \\ &+ (\delta_0^A \delta_B^0 - \delta_3^A \delta_B^3) [\delta_0^C (\delta_D^4 + \delta_D^5) + \delta_D^3 (\delta_4^C - \delta_5^C)] \\ &+ [\delta_0^A (\delta_B^4 + \delta_B^5) + \delta_B^3 (\delta_4^A - \delta_5^A)] (\delta_3^C \delta_D^3 - \delta_0^C \delta_D^0) \\ &+ (\delta_0^A \delta_B^1 + \delta_1^A \delta_B^3) [\delta_1^C (\delta_D^4 + \delta_D^5) - \delta_D^1 (\delta_4^C - \delta_5^C)] \\ &+ (\delta_0^A \delta_B^2 + \delta_2^A \delta_B^3) [\delta_2^C (\delta_D^4 + \delta_D^5) - \delta_D^2 (\delta_4^C - \delta_5^C)] \end{aligned} \quad (3.9a)$$

$$\begin{aligned} (R_2)^A{}_B{}^C{}_D &= (\delta_4^A - \delta_5^A) (\delta_B^4 + \delta_B^5) + \delta_0^A \delta_B^3 (\delta_4^C - \delta_5^C) (\delta_D^4 + \delta_D^5) \\ &- (\delta_4^A - \delta_5^A) \delta_B^3 \delta_0^C (\delta_D^4 + \delta_D^5) \\ &- 2\delta_0^A (\delta_B^4 + \delta_B^5) (\delta_4^C - \delta_5^C) \delta_D^3. \end{aligned} \quad (3.9b)$$

Using the basic relation of FRT formalism [24]

$$R^A{}_B{}^C{}_D G^B{}_E G^D{}_F = G^A{}_B G^C{}_D R^B{}_E{}^D{}_F, \quad (3.10)$$

we compute the commutation rules which determine the quantum conformal group algebra. This calculation is quite involved because, due to the complicated parametrization of $SO(4, 2)$ matrices, and we present only the relations for the parameters from the coset $\frac{SU(2, 2)}{SU(2; C)}$.

We get

$$[\hat{a}^\mu, \hat{a}^\nu] = \frac{i}{\kappa^2} (\delta_0^\nu \hat{a}^\mu - \delta_0^\mu \hat{a}^\nu) \quad (3.11a)$$

$$[e^{-\widehat{d}}, \widehat{a}^\mu] = 0 \quad (3.11b)$$

$$[e^{-\widehat{d}}, \widehat{c}^\mu] = \frac{2i}{\kappa} \left(\delta_0^\mu \widehat{c}_\alpha e^{-\widehat{d}} \widehat{c}^\alpha - \widehat{c}_0 e^{-\widehat{d}} \widehat{c}^\mu \right) - \frac{2}{\kappa^2} e^{-\widehat{d}} \widehat{c}^2 \widehat{c}_0 \delta_0^\mu \quad (3.11c)$$

$$[e^{-\widehat{d}} \widehat{c}^\mu, e^{-\widehat{d}} \widehat{c}^\nu] = \frac{i}{\kappa} e^{-\widehat{d}} \widehat{c}^2 e^{-\widehat{d}} (\delta_0^\alpha \widehat{c}^\mu - \delta_0^\mu \widehat{c}^\nu), \quad (3.11d)$$

and more complicated relation

$$\begin{aligned} [\widehat{a}^\mu, \widehat{c}_\nu] &= \frac{i}{\kappa} \left(\left(\widehat{c}^\alpha \widehat{\Lambda}^\mu_\alpha + 2 \widehat{c}^\alpha \widehat{a}^\mu e^{-\widehat{d}} \widehat{c}_\alpha \right) g_{0\nu} - \widehat{c}_0 \widehat{\Lambda}^\mu_\nu - 2 \widehat{c}_0 a^\mu e^{-\widehat{d}} \widehat{c}_\nu \right) e^{\widehat{d}} \\ &+ \frac{2}{\kappa^2} \left(\widehat{c}_\nu \widehat{\Lambda}^\mu_\beta \widehat{c}^\beta + \widehat{c}_0 \widehat{a}^\mu e^{-\widehat{d}} \widehat{c}^2 - e^{-\widehat{d}} \widehat{c}^2 \widehat{\Lambda}^\mu_0 \right. \\ &\quad \left. - 2 e^{-\widehat{d}} \widehat{c}^2 \widehat{a}^\mu e^{-\widehat{d}} \widehat{c}_0 \right) e^{\widehat{d}} g_{0\nu} \\ &- \widehat{a}^\mu \left(\frac{2i}{\kappa} \left(g_{0\nu} \widehat{c}_\alpha e^{-\widehat{d}} \widehat{c}^\alpha - \widehat{c}_0 e^{-\widehat{d}} \widehat{c}_\nu \right) \right. \\ &\quad \left. - \frac{2}{\kappa^2} e^{-\widehat{d}} \widehat{c}^2 \widehat{c}_0 g_{\nu 0} \right) e^{\widehat{d}}. \end{aligned} \quad (3.11e)$$

We see that the noncommutative translations a_μ form the algebra of κ -Minkowski space. Further the elements $e^{-\widehat{d}}$ and c_μ span another closed subalgebra. It can be rewritten in slightly more transparent form if we define

$$\widehat{s} \equiv e^{-\widehat{d}} \widehat{c}^2, \quad \widehat{l}^\mu \equiv e^{-\widehat{d}} \widehat{c}^\mu. \quad (3.12)$$

Then we obtain

$$\begin{aligned} [\widehat{s}, e^{-\widehat{d}}] &= 0 = [\widehat{s}, \widehat{l}^\mu], \\ [e^{-\widehat{d}}, \widehat{l}^\mu] &= \frac{2i}{\kappa} \left(\delta_0^\mu \widehat{l}^2 - \widehat{l}_0 \widehat{l}^\mu \right) - \frac{2}{\kappa^2} \widehat{s} \widehat{l}_0 \delta_0^\mu, \\ [\widehat{l}^\mu, \widehat{l}^\nu] &= \frac{i}{\kappa} \widehat{s} \left(\delta_0^\nu \widehat{l}^\mu - \delta_0^\mu \widehat{l}^\nu \right). \end{aligned} \quad (3.13)$$

The relations (3.13) we shall call κ -deformed conformal translations algebra.

In the relations (3.11a-3.11e) we describe the noncommutativity of quantum group parameters from the coset $\frac{O(4,2)}{O(3,1)} \simeq \frac{SU(2,2)}{SI(2;\mathbb{C})}$. By some additional calculational effort one could complete this list by including the noncommutativity relations involving quantum the parameters $\widehat{\Lambda}^A_B$.

Unfortunately, already set of relations (3.11a-3.11e) imply difficulties in introducing the structure of real quantum conformal group. It can be checked, that the hermicity properties of quantum conformal translations imply that $\widehat{c}_\mu \widehat{c}^\nu = 0$; this constraint is not invariant under the action of coproduct (3.6a). We conclude that our quantum conformal group can not be formulated as real quantum algebra.

4 Conclusions

It should be pointed out that defining noncommutative version of conformal symmetry is physically quite strongly motivated. Classical conformal symmetry implies that there is no geometrical scale (elementary length) which enters the conformal geometry describing conformal space-time structure. On other hand the notion of classical space-time seems to be restricted only to the distances larger than the Planck length (see e.g. [29]). Recently it is often conjectured (see e.g. [30]–[32]) that the Planck length parameter should enter the non-commutative geometry taking into consideration the quantum gravity effects. The quantum conformal symmetries with deformation parameter κ could be an example of such noncommutative symmetries which describe algebraically the quantum effects at the sub-Planck distances (in particular κ can be identified with the Planck mass).

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